INDIAN STATISTICAL INSTITUTE
Probability Theory II: B. Math (Hons.) I
Semester II, Academic Year 2016-17
Final Exam

Date: Apr 26, 2017 **Total Marks: 60** **Duration: 10:00 am - 1:00 pm**

- **Please write your roll number on top of your answer paper.**
- **Show all your works and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. A continuous random vector (X, Y) has a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} 2/3 & \text{if } x > 0, y > 0, x + y < 1, \\ c & \text{if } x < 1, y < 1, x + y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (2 marks) Find c.

   (b) (4 marks) Compute the conditional probability density function of Y given X.

   (c) (3+3 = 6 marks) Calculate $E(Y|X)$ and $Var(Y|X)$.

2. Suppose $(X_1, X_2, \ldots, X_{100}) \sim D(\alpha, \alpha, \ldots, \alpha; \beta)$ (the notation is as used in the class), where $\alpha \neq \beta$ are positive parameters.

   (a) (10 marks) Compute, with full justification, $E\left(\frac{\sum_{i=1}^{25} X_i}{\sum_{i=1}^{100} X_i}\right)$.

   (b) (2 marks) What will be your answer to Part (a) if $\alpha = \beta$?

3. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables having characteristic function $\phi(t) = \exp\{-|t|^{1.9}\}$, $t \in \mathbb{R}$ (assume that this is a valid characteristic function).

   (a) (6 marks) Express the cumulative distribution function of $S_n := X_1 + X_2 + \cdots + X_n$ $(n \geq 1)$ in terms of the cumulative distribution function of X_1.

   (b) (6 marks) Find the weak limit of $n^{-5/9} S_n$ as $n \to \infty$.

[P. T. O]

4. State whether each of the following statements is true or false. If it is true, give a detailed proof. On the other hand, if it is false, produce a counter-example with full justification. If you both prove and disprove a statement, you will get a zero in that problem.

   (a) (12 marks) $Z_n \xrightarrow{\mathbf{P}} Z$ if and only if $E\left(\frac{|Z_n - Z|}{1+|Z_n - Z|}\right) \to 0$ as $n \to \infty$.

   (b) (12 marks) Let $X_1, X_2, \ldots, X_n$ be i.i.d. standard normal random variables. For $k = 1, \ldots, n-1$, define $Y_k = (\sum_1^k X_i - kX_{k+1})/\sqrt{k(k+1)}$. Then $Y_1, \ldots, Y_{n-1}$ are also i.i.d. standard normal random variables.